Exact convergence rates of the last iterate in subgradient methods

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Contents

Subgradient methods

Subgradient methods

Objective: minimize a function $f : \mathbb{R}^d \to \mathbb{R}$ that is

- convex

  $$\partial f(x) = \{g \text{ such that } f(y) \geq f(x) + g^T(y - x) \text{ for all } y\} \neq \emptyset$$

- B-Lipschitz continous

  $$g \in \partial f(x) \Rightarrow \|g\| \leq B$$

- with minimizer x^*

Method: subgradient method with fixed step sizes $\{h_k\}$

$$x_{k+1} = x_k - h_k g_k \text{ for some } g_k \in \partial f(x_k)$$

starting from x_0

Performance criteria

Target: convergence rate after N iterations, either

- $\min_{0 \le k \le N} f(x_k) - f(x_*)$ (method is not monotone)
- $f(\frac{1}{N+1} \sum_{k=0}^{N} x_k) - f(x_*)$ (average)
- $f(x_N) - f(x_*)$

Initial iterate assumption:

$$\|x_0 - x_*\| \le R$$

Homogeneity: rates in function values must be proportional to BR

Lower bound: no method can achieve better rate than

$$\frac{BR}{\sqrt{N+1}}$$

Lower bound proof (variation of [Drori, Teboulle 2016])

Consider following function with $d = N + 1$, $B = 1$ and $x_* = 0$

$$f(x) = \max\{0, x_1, x_2, \ldots, x_{N+1}\} = \left[\max_{1 \le k \le N+1} x_k\right]_+$$

Choose starting point $x_0 = (1, 1, \ldots, 1)$ with $R = \sqrt{N+1}$

- As long as $f(x_k) > 0$, subgradient $g_k \in \partial f(x_k)$ can be chosen as a basis vector e_i for some $1 \le i \le N + 1$ (and $\|g_k\| = B$)
- Induction hypothesis (H_k) (easy to check for $k = 0$)
  x_k contains at least $N + 1 - k$ components equal to 1
- Assume (H_k) for $k \le N$. Then $f(x_k) \ge 1$. So subgradient g_k can be chosen as some basis vector e_i, and x_{k+1} can differ only by at most one component from x_k, implying (H_{k+1}) holds
- Conclusion:
$$f(x_k) \ \ge \ 1 = \frac{BR}{\sqrt{N+1}} \quad \text{for all } 0 \le k \le N$$

(also for other criteria / for steps with several past subgradients)

Standard convergence analysis

Only two ingredients:

(1) *subgradient inequality* and (2) *square distance telescoping*

Ingredient (1)

$$\begin{aligned}\|x^{k+1} - x^\star\|^2 &= \|x^k - h_k g^k - x^\star\|^2 \\ &= \|x^k - x^\star\|^2 + h_k^2\|g^k\|^2 - 2h_k\langle g^k, x^k - x^\star\rangle \\ &\leq \|x^k - x^\star\|^2 + h_k^2\|g^k\|^2 - 2h_k\left(f(x^k) - f(x^\star)\right).\end{aligned}$$

(where we have only used *subgradient inequality*
$f(x^\star) - f(x^k) \geq \langle g^k, x^\star - x^k\rangle$ between x^* and x^k)

This gives an upper bound on the accuracy $f(x^k) - f(x^*)$

$$h_k\big(f(x^k) - f(x^*)\big) \leq \tfrac{1}{2}\|x^k - x^*\|^2 - \tfrac{1}{2}\|x^{k+1} - x^*\|^2 + \tfrac{1}{2}h_k^2 B^2$$

using bound on subgradient norm $\|g_k\| \leq B$

Standard convergence analysis (cont.)

Ingredient (2) From

$$h_k\big(f(x^k) - f(x^*)\big) \leq \tfrac{1}{2}\|x^k - x^*\|^2 - \tfrac{1}{2}\|x^{k+1} - x^*\|^2 + \tfrac{1}{2}h_k^2 B^2$$

telescoping (summing from $k = 0$ to $k = N$) gives

$$\sum_{k=0}^{N} h_k\big(f(x^k) - f(x^*)\big) \leq \tfrac{1}{2}\left\|x^0 - x^\star\right\|^2 - \tfrac{1}{2}\|x^{N+1} - x^\star\|^2 + \tfrac{1}{2}B^2 \sum_{k=0}^{N} h_k^2$$

hence

$$\min_{0\leq k\leq N} f(x^k) - f(x^\star) \leq \frac{\tfrac{1}{2}\|x^0 - x^\star\|^2 + \tfrac{1}{2}B^2 \sum_{k=0}^{N} h_k^2}{\sum_{k=0}^{N} h_k}$$

Standard convergence analysis (end.)

$$\min_{0\le k\le N} f(x^k) - f(x^\star) \le \frac{\frac{1}{2}\|x^0 - x^\star\|^2 + \frac{1}{2}B^2 \sum_{k=0}^N h_k^2}{\sum_{k=0}^N h_k}$$

- Right-hand side is convex and symmetric in stepsizes h_k, hence optimal values are constant $h_k = h$ for all k

$$\min_{0\le k\le N} f(x^k) - f(x^\star) \le \frac{\frac{1}{2}\|x^0 - x^\star\|^2 + \frac{1}{2}B^2(N+1)h^2}{(N+1)h}$$

- Optimal h is then $h_k = \frac{R}{B}\frac{1}{\sqrt{N+1}}$ leading to an *optimal* rate

$$\min_{0\le k\le N} f(x^k) - f(x^\star) \le \frac{BR}{\sqrt{N+1}}$$

(same rate holds for average iterate since $f(\frac{1}{N+1}\sum_{k=0}^N x_k) \le \frac{1}{N+1}\sum_{k=0}^N f(x_k)$)

End of story?

What about last-iterate convergence?

$$\min_{0 \leq k \leq N} f(x^k) - f(x^\star) \leq \frac{BR}{\sqrt{N+1}}$$

- Says nothing about convergence of last iterate x_N
- O. Shamir, Open problem: Is averaging needed for strongly convex stochastic gradient descent? *JMLR* (2012)
- Practitioners often use the last iterate
- Storing best iterate might not be feasible (storage requirements, objective computation)
- Algorithm may correspond to a real-word dynamical system (see for example work by Nesterov and Shikhman)

Goal of this talk: study last-iterate convergence with and without performance estimation

Short history of our results

- 2012-2013: Drori and Teboulle introduce performance estimation problems (PEP)
  main idea: compute worst-case convergence rates

- 2013-2017: with Taylor and Hendrickx we further develop SDP-based PEP approach

- 2017: *Yurii* asks us "With your tool, can you tell the convergence rate of the last iterate in subgradient method?"

  We find a purely numerical rate (see next page), and no proof

- 2023: with Zamani we get back to the question and obtain a full PEP proof and a bit later a classic proof

Puzzle time

Puzzle: can you guess the convergence rate?

For constant stepsize $h = 1$ one can compute
using either PESTO (Matlab) or PEPIT (Python) toolboxes

$$f(x_N) - f(x_*) \leq BR\left[1 - N + \tfrac{1}{2}\left(s_N - s_N^{-1}\right)^2\right]$$

where the rate involves a mysterious sequence $\{s_k\}$:

$$s_0 = 1,\ s_1 = 2,\ s_2 = 2.5,\ s_3 = 2.9,$$

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$$s_0 = 1,\ s_1 = 2,\ s_2 = 2.5,\ s_3 = 2.9,\ s_4 = 3.24482758621,\ \ldots$$

or

$$s_0 = 1,\ s_1 = 2,\ s_2 = \frac{5}{2},\ s_3 = \frac{29}{10},$$

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$$s_0 = 1,\ s_1 = 2,\ s_2 = \frac{5}{2},\ s_3 = \frac{29}{10},\ s_4 = \frac{941}{290},\ \ldots$$

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Answer: $s_{k+1} = s_k + \frac{1}{s_k}$

This talk

Take-home messages:

- Performance estimation applied to subgradient methods
- Exact convergence rates can be obtained for the last iterate: suboptimal by a factor $O(\sqrt{\log(N)})$
- New last-iterate optimal method can be designed with linearly decreasing step sizes
- Extensions to constrained case, to normalized steps
- Inspiration for results provided by performance estimation but ultimately all proofs converted to classical style using a new key lemma

Contents

Last-iterate convergence

Tool: performance estimation

For a given PEP (Performance Estimation Problem) we can

- compute the exact value of the performance criteria's worst-case = optimal value of PEP problem
- identify an explicit function (and starting point) achieving this worst-case value = primal solution of PEP problem + interpolation
- obtain an independently-checkable proof that this worst-case value is a valid (upper) bound on the performance criteria = dual multiplier of PEP problem
- all three steps can be done either numerically or analytically

For a large class of first-order methods, including fixed-step subgradient methods, these can be computed *exactly* using a semidefinite programming (SDP) problem.

Interpolation conditions for nonsmooth convex functions

To perform PEP for subgradient methods on a class of functions

we need the corresponding *interpolation conditions* explicitly

given a list of values $(x_i, f_i, g_i)_{i \in I}$,
does there exist a convex f with B-bounded subgradients such that
$f(x_i) = f_i$ and $g_i \in \partial f(x_i)$ for all $i \in I = \{*, 0, 1, \dots N\}$

Necessary and sufficient conditions:

$$f(x_i) = f_i \text{ and } g_i \in \partial f(x_i) \text{ for every } i \in I$$

$$\Leftrightarrow$$

$$f_j \geq f_i + g_i^T (x_j - x_i) \text{ for every } i, j \in I$$

$$\|g_i\| \leq B \text{ for every } i \in I$$

Leads to a convex, tractable formulation as a SDP

Results: average iterate with constant stepsize

Worst-case for constant stepsize subgradient method

$$x_{i+1} = x_i - h(\tfrac{R}{B})g_i$$

applied to convex function with B-bounded subgradients

- For *average* value of iterates $\hat{f}_N = \frac{f(x_0)+f(x_1)+\ldots+f(x_N)}{N+1}$, tight worst-case is

$$\hat{f}_N - f(x_*) \leq \begin{cases} BR\left(\frac{1}{2}h + \frac{1}{2(N+1)}\frac{1}{h}\right) & \text{when } h \geq \frac{1}{N+1} \\ BR\left(1 - \frac{N}{2}h\right) & \text{when } h \leq \frac{1}{N+1} \end{cases}$$

  (recovers result shown earlier for large h)

- Optimal constant step-size is then $h^* = \frac{1}{\sqrt{N+1}}$ (belongs to "large step" case) leading to tight worst-case

$$\hat{f}_N - f(x_*) \leq \frac{BR}{\sqrt{N+1}}$$

Results: last iterate with constant stepsize

- Define sequence $\{s_i\}_{i\geq 0} = \{1, 2, \frac{5}{2}, \frac{29}{10}, \ldots\}$ with
$$s_0 = 1, s_{i+1} = s_i + \frac{1}{s_i} \text{ for all } i \geq 0$$
- No closed form but s_N^2 grows like $2(N+1) + \frac{1}{2}\log(N)$, also appears in [Nesterov 2009] (again!) for primal-dual subgradient
- For value of *last* iterate $f(x_N)$, tight worst-case is
$$f(x_N) - f(x_*) \leq \begin{cases} BR\left[(\frac{1}{2}s_N^2 - N)h + \frac{1}{2s_N^2}\frac{1}{h}\right] & \text{when } h \geq \frac{1}{s_N^2} \\ BR(1 - Nh) & \text{when } h \leq \frac{1}{s_N^2} \end{cases}$$
- No previous result with correct asymptotic rate for last iterate
- [Harvey,Liaw,Plan,Randhawa 2019] prove a $\frac{\log N}{32\sqrt{N}}$ lower bound when $B = 1$ with stepsize $h_i = \frac{1}{\sqrt{i}}$, and prove a high probability $\mathcal{O}(\frac{\log N}{\sqrt{N}})$ upper bound in stochastic case

Results: optimal stepsize and variants

- To perform N subgradient iterations, optimal stepsize is then

$$h^* = \frac{1}{s_N \sqrt{s_N^2 - 2N}}$$

and corresponding exact worst-case convergence rate becomes

$$f(x_N) - f(x_*) \leq BR \sqrt{1 - \frac{2N}{s_N^2}} \lesssim BR \cdot \sqrt{\frac{1 + \frac{1}{4}\log(N)}{N+1}}$$

- Using $h = \frac{1}{\sqrt{N+1}}$ (now known to be suboptimal for last iterate) leads to slightly worse

$$f(x_N) - f(x_*) \leq BR \cdot \left(\frac{\frac{5}{4} + \frac{1}{4}\log(N)}{\sqrt{N+1}} \right)$$

Results were obtained using the following PEP

$$
\begin{aligned}
\max\ & f^{N+1} - f^\star \\
\text{s.t. } & f^i - f^j - \left\langle \tfrac{B}{Rh}(x^j - x^{j+1}), x^i - x^j \right\rangle \geq 0 \quad i \in \{1, \dots, N+1, \star\}, j \in \{1, \dots, N\} \\
& f^i - f^{N+1} - \left\langle g^{N+1}, x^i - x^{N+1} \right\rangle \geq 0 \quad i \in \{1, \dots, N+1, \star\} \\
& f^i - f^\star \geq 0 \quad i \in \{1, \dots, N+1\} \\
& R^2 h^2 - \left\| x^k - x^{k+1} \right\|^2 \geq 0 \qquad k \in \{1, \dots, N\} \\
& B^2 - \left\| g^{N+1} \right\|^2 \geq 0 \\
& R^2 - \left\| x^1 - x^\star \right\|^2 \geq 0.
\end{aligned}
$$

PEP-based proof is ... straightforward?

Define $f^i = f(x^i)$ and $\sigma_i = \frac{1}{s_{i+1}}, \quad i \in \{0, 1, ..., N\}$ and observe that

$$
\begin{aligned}
& f^{N+1} - BR\left(\left(\tfrac{1}{2}s_{N+1}^2 - N\right)h + \tfrac{1}{2s_{N+1}^2 h}\right) + \sum_{i=1}^{N} \tfrac{B\sigma_{N-i}^2}{2Rh}\left(R^2h^2 - \left\|x^i - x^{i+1}\right\|^2\right) \\
& + \sum_{i=1}^{N}\sum_{j=i+1}^{N} \sigma_{N-j}\left(\sigma_{N-i} - \sigma_{N+1-i}\right)\left(f^i - f^j - \left\langle \tfrac{B}{Rh}(x^j - x^{j+1}), x^i - x^j\right\rangle\right) \\
& + \sum_{i=1}^{N}\left(\sigma_{N-i} - \sigma_{N+1-i}\right)\left(f^i - f^{N+1} - \left\langle g^{N+1}, x^i - x^{N+1}\right\rangle\right) + \tfrac{B\sigma_N^2}{2Rh}\left(R^2 - \left\|x^1\right\|^2\right) \\
& + \sigma_N \sum_{i=1}^{N} \sigma_{N-i}\left(-f^i - \left\langle \tfrac{B}{Rh}(x^i - x^{i+1}), -x^i\right\rangle\right) + \tfrac{Rh}{2B}\left(B^2 - \left\|g^{N+1}\right\|^2\right) \\
& + \sigma_N\left(-f^{N+1} + \left\langle g^{N+1}, x^i\right\rangle\right) \\
= & \tfrac{-Rh}{2B}\left\|g^{N+1} - \tfrac{B}{Rh}x^{N+1} + \tfrac{B}{Rh}\sum_{i=1}^{N}\left(\sigma_{N-i} - \sigma_{N+1-i}\right)x^i\right\|^2 \leq 0.
\end{aligned}
$$

Post-PEP reflections

- After staring at the PEP proof, we noticed similarities between inequality multipliers
- Grouping similar terms, we obtain Jensen-like inequalities (insight: applying Jensen $\leftrightarrow$ some sum of interpolation inequalities)
- Simplifying further we obtain a classic-style proof, that is no longer looking computer generated
- We encapsulate the main part of the proof in a *key Lemma*
- Key Lemma fully reverse-engineered from PEP but can be easily checked by hand

Key Lemma for subgradient methods

Lemma ([Zamani,G 2023])

Consider the subgradient method with fixed step sizes $\{h_k\}$

$x_{k+1} = x_k - h_k g_k$ *for some* $g_k \in \partial f(x_k)$ *for* $k = 0, 1, \ldots, N-1$

Choose $h_N > 0$ *and introduce* $N+2$ *weights* v_k *that satisfy*

$$1 = v_{-1} \leq v_0 \leq v_1 \leq \cdots \leq v_{N-1} \leq v_N$$

Then iterates after N *iterations of the subgradient method satisfy*

$$\sum_{k=0}^{N} \left(h_k v_k^2 - (v_k - v_{k-1}) \sum_{i=k}^{N} h_i v_i \right) \left(f(x^k) - f(x^*) \right)$$

$$\leq \frac{1}{2} \underbrace{\left\| x^0 - x^* \right\|^2}_{R} + \frac{1}{2} \sum_{k=0}^{N} h_k^2 v_k^2 \underbrace{\left\| g^k \right\|^2}_{B}$$

Why is the key Lemma useful?

Key Lemma inequality:

$$\sum_{k=0}^{N}\left(h_k v_k^2-(v_k - v_{k-1})\sum_{i=k}^{N} h_i v_i\right)\left(f(x^k)-f(x^*)\right) \leq \tfrac{1}{2}v_0^2 R^2+\tfrac{1}{2}B^2\sum_{k=0}^{N} h_k^2 v_k^2$$

for any weights $1 = v_{-1} \leq v_0 \leq v_1 \leq \cdots \leq v_{N-1} \leq v_N$

- constant $v_k = 1$ recovers usual (average) rate
- but a suitable choice of $\{v_k\}$ allows us to modify coefficients in front of $f(x^k) - f(x^*)$
- in particular one can cancel all coefficients except last one in front of $f(x^N) - f(x^*)$

Idea of the proof of the key Lemma

Inequality to prove:

$$\sum_{k=0}^{N}\Big(h_k v_k^2-(v_k-v_{k-1})\sum_{i=k}^{N} h_i v_i\Big)\big(f(x^k)-f(x^*)\big)\le \tfrac{1}{2}v_0^2R^2+\tfrac{1}{2}B^2\sum_{k=0}^{N}h_k^2v_k^2$$

Proof uses a *generalization* of the standard telescoping proof

1. From weights v_k define *auxiliary sequence* z^k recursively

$$z^0=x^* \quad \text{and} \quad z^k=\big(1-\frac{v_{k-1}}{v_k}\big)x^k+\big(\frac{v_{k-1}}{v_k}\big)z^{k-1}$$

This implies

$$z^k=\big(\frac{v_0}{v_k}\big)x^*+\sum_{i=1}^{k}\big(\frac{v_i-v_{i-1}}{v_k}\big)x^i$$

(note z^k is a convex combination of x^* and iterates x^i)

Idea of the proof of the key Lemma (cont.)

2. Subgradient inequality between x^k and z^k (instead of x^*) gives

$$h_k v_k^2 \big(f(x^k) - f(z^k)\big) \leq \tfrac{1}{2} v_{k-1}^2 \|x^k - z^k\|^2 - \tfrac{1}{2} v_k^2 \|x^{k+1} - z^{k+1}\|^2 + \tfrac{1}{2} B^2 h_k^2 v_k^2$$

3. Telescoping (summing from $k = 0$ to $k = N$) gives that

$$\sum_{k=0}^{N} h_k v_k^2 \big(f(x^k) - f(z^k)\big)$$

$$\leq \quad \tfrac{1}{2} v_{-1}^2 \|x^0 - z^0\|^2 - \tfrac{1}{2} v_N^2 \|x^{N+1} - z^{N+1}\|^2 + \tfrac{1}{2} B^2 \sum_{k=0}^{N} h_k^2 v_k^2$$

implying

$$\sum_{k=0}^{N} h_k v_k^2 \big(f(x^k) - f(z^k)\big) \leq \tfrac{1}{2} \|x^0 - x^*\|^2 + \tfrac{1}{2} B^2 \sum_{k=0}^{N} h_k^2 v_k^2$$

Idea of the proof of the key Lemma (cont.)

4. Finally we need to find a lower bound on $f(x^k) - f(z^k)$ terms:

$$z^k = \left(\frac{v_0}{v_k}\right)\hat{x} + \sum_{i=1}^{k}\left(\frac{v_i - v_{i-1}}{v_k}\right)x^i$$

implies, by Jensen's inequality

$$f(z^k) \leq \left(\frac{v_0}{v_k}\right)f(\hat{x}) + \sum_{i=1}^{k}\left(\frac{v_i - v_{i-1}}{v_k}\right)f(x^i)$$

hence

$$h_k v_k^2\big(f(z^k) - f(x^*)\big) \geq h_k v_k \sum_{i=1}^{k}(v_i - v_{i-1})\big(f(x^i) - f(x^*)\big)$$

which combined with inequality from the previous step 3. gives

$$\begin{aligned} & \sum_{k=0}^{N}\left(h_k v_k^2 - (v_k - v_{k-1})\sum_{i=k}^{N} h_i v_i\right)\Big(f(x^k) - f(x^*)\Big) \\ \leq & \sum_{k=0}^{N} h_k v_k^2\big(f(x^k) - f(z^k)\big) \leq \tfrac{1}{2}\|x^0 - x^*\|^2 + \tfrac{1}{2}B^2\sum_{k=0}^{N} h_k^2 v_k^2 \end{aligned}$$

Using the key Lemma

So we have proved

Lemma
Iterates of the subgradient methods satisfy

$$\sum_{k=0}^{N} \left(h_k v_k^2 - (v_k - v_{k-1}) \sum_{i=k}^{N} h_i v_i \right) \left(f(x^k) - f(x^*) \right)$$
$$\leq \tfrac{1}{2} R^2 + \tfrac{1}{2} B^2 \sum_{k=0}^{N} h_k^2 v_k^2$$

Proof of last-iterate convergence rate:

Choose weights v_k that cancel all coefficients of $f(x^k)$ except $f(x^N)$, which are

$$v_k = \frac{1}{s_{N+1-k}}$$

Exactness of convergence rate

All PEP rates are *exact* by design

(cannot be improved, even by a multiplicative/additive constant)

Follows from PEP solution, but can be made constructive by building an *explicit worst-case function*

- Function of the type $f(x) = [\max_k \{g_k^T x\}]_+$
- Recursive definition, coefficients g_k not straightforward

- Sugbradients for all iterates are g_k, have maximum norm B
- Subgradient inequality is satisfied between all pairs of iterates
- Matches exactly the announced convergence rate for the last iterate

Extensions

Last-iterate optimal subgradient method

Define the following new linearly decreasing stepsize schedule

$$x_{k+1} = x_k - \frac{R}{B}\frac{(N+1-k)}{(N+1)^{3/2}} g_k$$

Leads the optimal rate for the last iterate [Zamani,G 2023]

$$f(x_N) - f(x_*) \leq \frac{BR}{\sqrt{N+1}}$$

- Improves $\frac{15BD}{\sqrt{N+1}}$ [Jain,Nagaraj,Netrapalli 2021] for diameter D
- Same proof technique, key lemma with optimized weights v_k
- *Schedule dependence on N* is forced for optimal method (already impossible to find fixed stepsizes h_1 and h_2 that are optimal for both $N = 1$ and $N = 2$)
- *Open question*:
  Existence of a last-iterate optimal method with stepsizes independent from N and with momentum terms?

Subgradient method with normalized step sizes

Stepsizes so far feature a $\frac{R}{B}$ factor, require knowledge of R and B

- constant stepsizes $h_k = \frac{R}{B} h$ for some h
- optimal stepsizes $h_k = \frac{R}{B} \frac{(N+1-k)}{(N+1)^{3/2}}$

Need for B can be removed using normalized step sizes $\{t_k\}$

$$x_{k+1} = x_k - t_k \frac{g_k}{\|g_k\|} \text{ for some } g_k \in \partial f(x_k)$$

- All previous results are also valid with exactly the same rates if we assume $t_k = h_k B$
- constant stepsizes $t_k = Rh$ for some h
- optimal stepsizes $t_k = R \frac{(N+1-k)}{(N+1)^{3/2}}$
- Proof using key Lemma with adapted weights
- Removing dependence on R seems harder ($\rightarrow$ parameter-free)

Projected subgradient method

Solve convex constrained optimization

$$\min_{x \in X} f(x)$$

with the projected subgradient method with fixed step sizes $\{h_k\}$

$$x_{k+1} = \mathbb{P}\left[x_k - h_k g_k\right] \text{ for some } g_k \in \partial f(x_k)$$

($\mathbb{P}$ is orthogonal projection on convex set X)

- All results are also valid, with exactly the same rates (both constant and optimal stepsizes, also normalized)
- Straightforward adaptation of the key Lemma using non-expansiveness of the projection operator

Conclusions

Conclusions

Take-home messages:

- Performance estimation applied to subgradient methods
- Exact convergence rates can be obtained for the last iterate: suboptimal by a factor $O(\sqrt{\log(N)})$
- New last-iterate optimal method can be designed with linearly decreasing step sizes
- Extensions to constrained case, to normalized steps
- Inspiration for results provided by performance estimation but ultimately all proofs converted to classical style using a new key lemma

For all your performance estimation needs:

`https://github.com/PerformanceEstimation`

Thank you Yurii!